ON THE GENERALIZED MINIMUM COST FLOW PROBLEM: AN APPLICATION IN NATURAL GAS DISTRIBUTION NETWORKS

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Abstract

We consider an extension of the minimum cost flow problem (MCFP) in a network where each edge (i,j) (arc or link) has a multiplier $\alpha(i,j)$. Such problems occur in network based systems where flow is not conserved on every edge. We consider this problem for the natural gas distribution networks with edges representing pipelines and compressors (multipliers) and nodes connects pipelines together or reroutes pipelines. First, we give an analysis of the problem in gas distribution networks from a graph point of view then provide a solution using an extension of the successive shortest path algorithm which is illustrated with a numerical example of interest consisting of 7 nodes, 9 pipelines and 2 compression stations with data chosen arbitrarily. Results indicated that flow is not conserved as gas flow at the sink node is reduced compared to that at the start node

Keywords: Generalized minimum cost flow; Successive shortest path; Natural gas distribution networks.

Introduction

We had interactions or conversations with individuals or groups of people or physical entities can be likened to a web of connections or the familiar word network. Thus we may encounter social networks, communication networks, telephone networks etc in our everyday exploits. We make use of networks in our everyday lives, the telephone networks helps us to communicate with people around the world, road, rail and airline networks enables us to travel across great distances, manufacturing and distribution networks gives us easy access to important products. In all these, networks helps with our desire to efficiently transfer flow (food, products, fluids, people etc.) from one place to another. Our focus is on the natural gas distribution networks which is one of the final steps in delivering gas to its end users or customers.

Problems that one may face when transferring flow or products from one point or location to another in a network based systems are generally called network flow problems. Network flow problems are types of optimization problems and can be used in modelling assignment, chemical, distribution and other processes (Aderibigbe and Apanapudor, 2014), (Park, 2015). Some of these problems are the maximum value problem which involves finding the maximum amount of flow that can be sent from the start to the end nodes, shortest path problem involves finding the shortest or cheapest route between two nodes etc. The study and analysis of network based systems and their flow is very important in modelling and providing optimal solutions to these problems (Ahuja *et al.*, 1993), Apanapudor and Izevbizua (2018).

Graph theory provides algorithms for solving problems in network based systems. Graph theory generally involves points (nodes) joined together or connected by lines (edges), it is simply a graphical representation of the relationship or connections between objects. Mathematically, we can represent a network G by:

$$G = (V, E)$$

Graphically

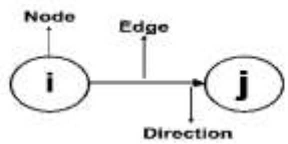

Figure 1 Simple Flow Network.

Where V is the set of nodes or vertices i, j and E is the set of edges or links, i.e. $i, j \in V$ and $(i, j) \in E$.

Graph theory is the study of sets of linked nodes (Rodrigue and Ducruet, 2022). These objects (nodes and edges) can be used as abstracts for different network based systems and are used to model many problems in mathematics, computer science, and engineering. In this paper, our focus is on the natural gas distribution networks which can be carried out both on land and water, (Okwonu and Apanapudor, 2019), (Izevbizua and Apanapudor, 2019)

Distribution of gas is usually one of the final steps in delivering natural gas to customers, while it is sometimes distributed directly from pipelines(interstate or intrastate) to commercial and industrial customers through marketing companies, for others, they receive gas from local gas utility. Gas distribution networks are made up of pipelines that connects a gas consumer or customer from a source. The consumer or the customer has a role to play in quantity being ordered to supplied or distributed, Izevbizua and Apanapudor(2022). This can easily be represented by the network

$$G = (V, E)$$

In this case V is the set of nodes signifying where two or more pipelines are connected, where pipelines are rerouted, source location of the gas and gas customer(s), while E is the set of pipelines.

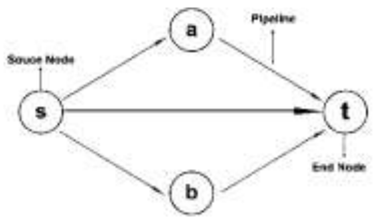

Figure 2 A typical natural gas distribution network

Natural gas networks covers all transmission and distribution pipelines that links natural gas from production areas and storage facilities with consumers or end users (Farzaneh-Gord, 2016). These networks are systems with hundreds or thousands of kilometres of pipelines, distribution centers, compression stations, regulators and valves that helps deliver gas to consumers (Gonzalez, 2009).

Generalized flow problems are problems associated with a generalized network. In a generalized network, each edge has a positive flow multiplier associated with it (Cohen & Megiddo, 1993). This multiplier is said to be loss if its< 1, gain if its > 1 and results to normal traditional network flow problems if its $= 1$ (Ahuja et al., 1993).

The rest of this paper is organized as follows. Section two dwell on materials and methods employed to drive this work to our desire. Section three focus on the analysis, results and discussion, while four dwelt on the the conclusion and section five discussed areas for further studies.

Materials and Methods

The minimum cost flow problem involves finding the least cost of sending flow through a network from its source to end nodes Apanapudor (2019). The problem is defined as:

$$minimize \sum_{(p,q)\in E} k(p,q)f(p,q)$$

Subject to

$$\sum_{\{q:(p,q)\in E\}} f(p,q) - \sum_{\{q:(q,p)\in E\}} f(q,p) = d_o$$

where for each edge $(p,q) \in E$ and nodes $p,q \in V$, $k(p,q)$ is cost of shipment through that edge and $f(p,q)$ is amount of product, d_o is the node potential. In traditional flow networks, there is an assumption that flow is conserved on every edge. That is for each edge (i,j), the amount of flow leaving node i equals the amount entering j. This assumption is reasonable in some applications. For example if a car company is distributing cars to its warehouses, it is reasonable to assume that the amount of cars is conserved. This assumption may be violated in some applications.

In this paper, our focus is on the natural gas distribution networks. Distribution of such commodity is essential for the wellbeing of the consumer, Tsetimi, et al(2021). While it is reasonable to assume that gas flow is conserved for each pipeline since demands are satisfied, in practice, this assumption may not be true due to the nature of the gas and its network. For example, in situations where there is leakage as gas is being transported through a pipeline say (i,j), the gas flow $f(i,j)$ that enters (i,j) through i will be less than the flow that arrives at j. In addition to minimum cost flow problem, we introduce a positive multiplier $\alpha(p,q)$ for each pipeline, indicating that if gas flow $f(p,q)$ enters node i, then $\alpha(p,q)f(p,q)$ enters node j. Defining the problem:

$$minimize \sum_{(p,q)\in E} k(p,q)f(p,q)$$

Subjected to
$\sum_{\{q:(p,q)\in E\}} f(p,q) - \sum_{\{q:(q,p)\in E\}} f(q,p)\,\alpha(p,q) =$
d_oMethods

Flow Network

We define the network G with a directed graph:

$$G = (V, E, f, c, k, \alpha)$$

where E is the set of pipelines, V is the set of nodes connecting the pipelines, f is the gas flow, c is the capacity of the pipeline, k is the cost of shipment trough each pipeline, α is a multiplier attached to each pipeline such that for each unit of gas flow $f(i,j)$ that enters pipeline (i,j) having a compression station or leak at node i, only $\alpha(i,j)f(i,j)$ flow arrive at node j. That is:

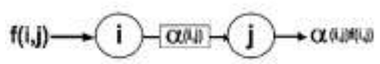

Figure 3 Flow network with compressor.

Multiplier α

Due to the nature of gas and its networks, the traditional conservation of flow for each edge is often violated based on limited tolerance on quantity and quality to maintain end-users' demand. Compressor stations are along the networks to restore pressure dropped along the line as a result of friction thus reducing its volume and correcting its flow speed (Fluenta, 2019).These stations are usually fuelled by a portion of the gas being transported consuming about 3 to 5% of it (Eishiekh, 2014). The goal is to solve the minimum cost flow problem with the above assumption. Going by this if a gas flow say $50\,cf$ is transported through a pipeline (i,j) with compressor station that uses 4% of the transported gas to power this station, thus only $48\ cf$ of gas is expected to arrive at node j. Mathematically:

$$(100-4)\%\ of\ 50\ cf$$

$$= \frac{96}{100} \times 50 = 48$$

$$\Rightarrow \alpha(i,j) = \frac{96}{100} = \frac{24}{25}$$

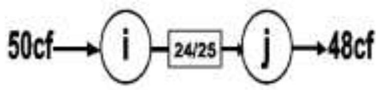

Figure 4 Flow network with compressor.

Cavalieri (2017) proposed a novel complete steady–state flow formulation in governing nonlinear system of equations and expression of the error function to be minimized while locating the solution.

Node Potentials

For each node i, the node potential d_i is equal to the cost or length of the cheapest or shortest paths from $s \; to \; t$ Sedlock (1985), Rosen (1998). The node potential tells the path with the least cost of shipment from the source node s to the end node t.

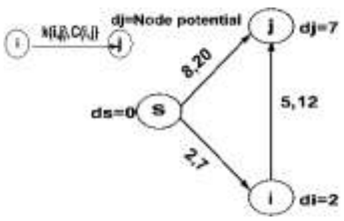

Figure 5 Network showing node potentials

The start node is given the potential value of 0, since there have been no movement. Other node's potentials are updated with respect to the shortest or cheapest path from s. From figure 5,

$$d_s = 0$$
$$d_i = d_s + k(s,i) = 0 + 2$$
$$\Rightarrow d_i = 2$$
$$d_j = d_s + k(s,j); d_i + k(i,j) = 0 + 8; 2 + 5$$
$$d_j = 8; 7 \; but \; 8 > 7$$
$$Thus \quad d_j = 7$$

Reduced Cost

In order to keep each pipeline's cost nonnegative, we calculate its reduced cost after every iteration using the updated node potentials. Mathematically, from figure 5

$$k(s,i) \leftarrow k(s,i) + d_i - d_j$$

$$k(s,i) \leftarrow 2 + 0 - 2 = 0$$

$$k(s,j) = 1$$

$$k(i,j) = 0$$

Residual Network

This is a network showing unused pipeline capacities. It is constructed using the residual capacity c_r.

$$c_r(i,j) = c(i,j) - f(i,j)$$

From figure 5, if 15 unit of gas flow is sent from through $s \rightarrow t$, the residual network will have a residual capacity of 5, since

$$c_r(s,j) = c(s,j) - f(s,j)$$

$$c_r(s,j) = 20 - 15 = 5$$

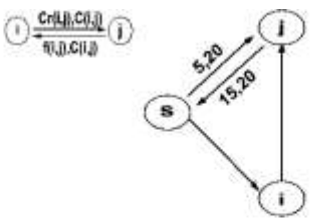

Figure 6 Simple Residual Network

From figure 6 we observe that (s,j) is replaced with (s,j) and (j,s). The forward arrow (s,j) indicates the amount of unused pipeline capacity. The backward arrow (j,s) indicates the amount of used capacity and flow that can be sent back.

Proposed Algorithm for The Generalized Flow Problem

This algorithm is achieved by extending the successive shortest path algorithm to accommodate α (changes in flow). Just as in the successive shortest path algorithm, we optimize flow and paths simultaneously. We send gas flow from $s \; to \; t$ along the shortest path (with respect to the pipeline's optimal paths), update the residual network Apanapudor and Izevbizua (2018). We look for another available path to send flow etc. The algorithm terminates when there are no paths from $s \; to \; t$ in the residual network.

i. Initialize $f = 0$ across the network.
ii. While (the residual network G_r contains a path from node $s \; to \; t$) do
iii. Find node potentials and calculate reduced costs using node potentials
iv. Find cheapest path (with respect to the pipeline's cost of shipment) from $s \; to \; t$
v. Find edge with least capacity c along path.
vi. If (Path has n numbers of multipliers α on edges)
vii. Send flow $f = (\frac{1}{\alpha_1 \cdot \alpha_2 \cdots \alpha_n}) \times c \leq c$ along this path
viii. Update the residual network G_r

Analysis, results and discussion

In order to demonstrate the key moment of the optimization method, we give the solution of an illustrative example of a gas network containing 7 nodes, 9 pipelines, 2 compression stations on edges $(s,a) \; and \; (p,j)$ both using 5% of the distributed gas for its operation.

Such data gathered would enable us do a robust analysis of our methods, which would go a long way to improve the performance of our system and management decision processes, Okwonu, *et al.*, (2022)

Given the network

$$G = (V, E)$$

Where $V = \{s, a, i, j, p, q, t\}$ the set of nodes is

$E = \{(s,a), (a,i), (a,p), (i,p), (i,j), (p,j), (p,q), (j,t), (q,t)\}$ is the set of edges.

The multiplier α of some edges indicating compressors are given in rectangles.

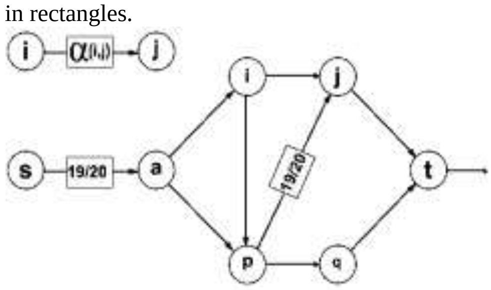

Figure 7 Nodes, edges (pipelines), multiplier (compressor)

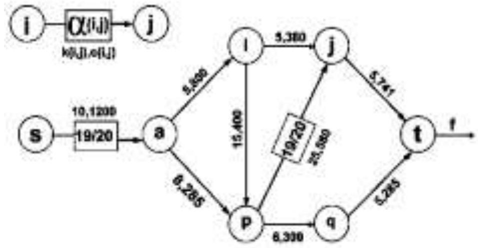

Figure 8 Flow Network

First, we initialize a flow of 0 across network and calculate each node's potential using the cost of shipment.

$$d_s = 0; d_a = d_s + c(s,a) = 10, c(s,a) = 10$$

$$d_i = d_s + d_a + c(a,i) = 15$$

$$d_p = 18, d_q = 24\,, d_j = 20, d_t = 25$$

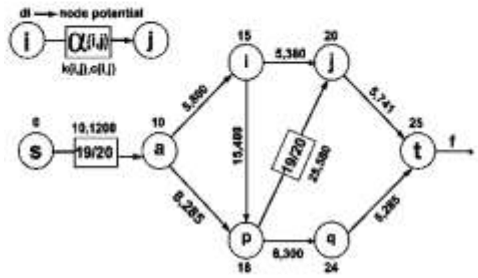

Figure 9 Flow network with node potentials

Next we calculate the reduced cost for each edge using the node potentials

$$k(p,q) \leftarrow k(p,q) + d_p - d_q,$$

where $(p,q) \in E\,;\ \ p,q \in V$

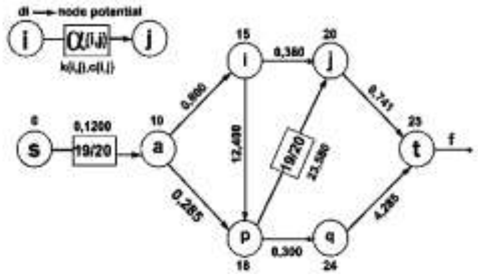

Figure 10 Node potentials and reduced costs

The first optimal path is $\boldsymbol{s \to a \to i \to j \to t}$. The edge with least capacity is $(i,j) = 380$. Instead of sending 380 as usual, we send

$$\frac{1}{\alpha} \times 380 = \frac{1}{\frac{19}{20}} \times 380$$

$$f = 400$$

Thus, we send 400 flow through this path.

After going through (s,a), only 380 is available for node i

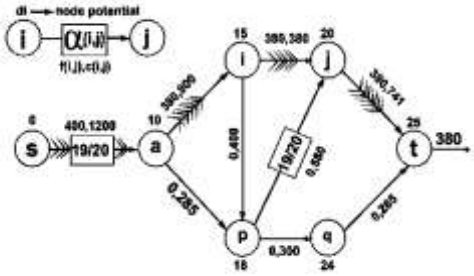

Figure 11 1st optimal path $\boldsymbol{s \to a \to i \to j \to t}$

The corresponding residual network and new node potentials.

$$d_s = 0;\ d_a = 0\ ;\ d_i = 0;\ d_p = 12$$

$$d_j = d_p + c(p,j) = 12$$

$$d_q = 12\ ;\ d_t = 12$$

Residual capacity

$$c_r(p,q) = c(p,q) - f(p,q)$$

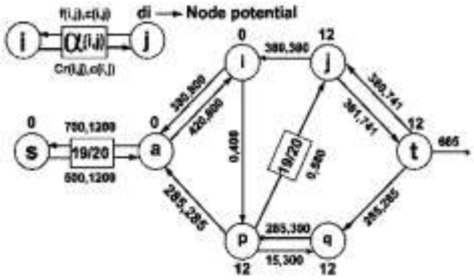

Figure 12 Residual network with new node potentials

Reduced cost for each edge using new node potentials

$$k(s,a)0\,; k(a,i) = 0, k(a,p) = 0; k(i,j) = 0,$$

$$k(i,p) = 12\,;\ \ k(p,q) = 0$$

$$k(p,j) = 0\,;\ \ k(j,t) = 19\,;\ k(q,t) = 0$$

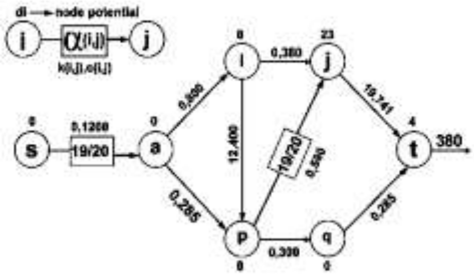

Figure 13 Reduced Cost

The second optimal path is $\boldsymbol{s \to a \to p \to q \to t}$. The edge with least capacity is $(a,p) = (i,j) = 285$. Instead of sending 285 as usual, we send

$$\frac{1}{\alpha} \times 380 = \frac{1}{\frac{19}{20}} \times 285$$

$$f = 300$$

Thus, we send 300 flow through this path.

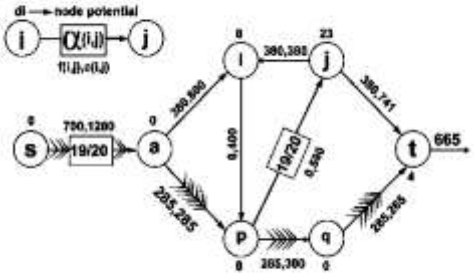

Figure 14 2nd^t optimal path $s \rightarrow a \rightarrow p \rightarrow q \rightarrow t$

The corresponding residual network and new node potentials.

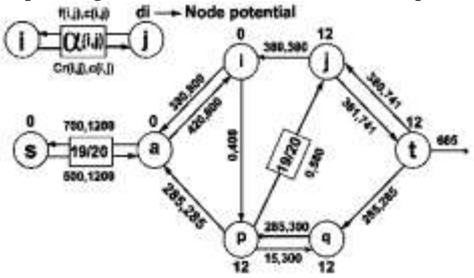

Figure 15 Residual network with new node potential

Reduced cost for each edge using new node potentials

$$k(s,a) = 0\ ;\ k(a,i) = 0,$$
$$k(a,p) = 0\ ;\ k(i,j) = 0,$$
$$k(i,p) = 0\ ;\ \ k(p,q) = 0$$
$$k(p,j) = 0\ ;\ \ k(j,t) = 0$$
$$k(q,t) = 0$$

The third and only path left is $\boldsymbol{s \rightarrow a \rightarrow i \rightarrow p \rightarrow j \rightarrow t}$. The edge with least capacity is $= (j,t) = 361$ We send

$$\frac{1}{\alpha_1} \cdot \frac{1}{\alpha_2} \times 361 = \frac{1}{\frac{19}{20}} \times \frac{1}{\frac{19}{20}} \times 361$$
$$f = 400$$

Thus, we send 400 flow through this path

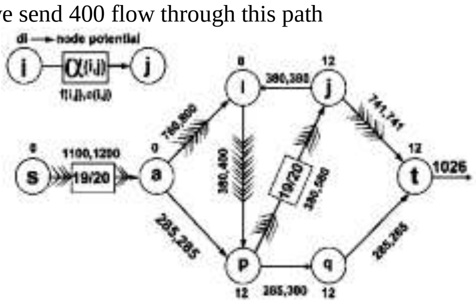

Figure 16 3rd^t optimal path $s \rightarrow a \rightarrow i \rightarrow p \rightarrow j \rightarrow t$

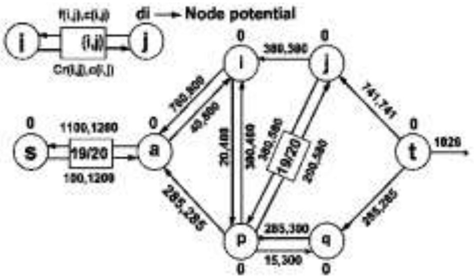

Figure 17 Final Residual network

Since both (j,t) and (q,t) can't take any more flow, the algorithm ends here.
After all iterations the final flow network will be:

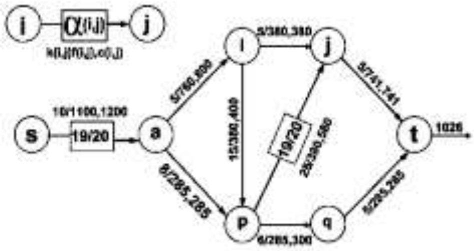

Figure 18 Flow Network

Calculating the objective function:

$$\sum_{(p,q)\in E} k(p,q)f(p,q)$$

$=(10 \times 1110) + (770 \times 5) + (285 \times 8) + (380 \times 5) + (285 \times 6)$

$+(390 \times 25) + (750 \times 5) + (285 \times 5)$

$= 41615$

Conclusion

Natural gas networks are systems with hundreds or thousands of kilometers of pipelines, production, storage and distribution centers, compression stations, and many other devices like valves and regulators, which when put together helps with gas distribution. Due to the nature of these networks, the assumption of conservation of flow on every edge is violated.

From our example, the inclusion of compressor stations along (s,a) and (p,j) makes the above assumption false as flow is not conserved. From figure 18, observe that a total of 1110 units of gas is delivered into the network at s but only 1026 arrived at its destination t. From our calculations, both compressor stations along (s,a) and (p,j) uses 5% the transported gas for its running and operation. The above algorithm optimizes both flow and cost of shipment simultaneously. Finally, Figure 11 gives the optimal path from s to t which is $\boldsymbol{s \rightarrow a \rightarrow i \rightarrow j \rightarrow t}$ Depending on the length or cost of shipment. From figure 18, 1026 units of gas is the maximum amount of flow that can move from s to t per unit time.

Recommendation for further studies

In this paper, the Successive shortest path algorithm of graph theory is extended to solve a generalized form of the minimum cost flow problem in natural gas networks. The algorithm can be used to model and solve important flow problems in network based systems where flow is not conserved as its being distributed for example the distribution of volatile gas that is susceptible to evaporation, transportation of raw crude, where flow might be lost due to leaks, the financial sector, where currencies have different values and rates. Based on the algorithm, detailed analysis and calculation procedure, one can create computer codes using non-negative real weighted values from actual information and data.

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